# The $\eta'$ in baryon chiral perturbation theory[1]

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**Abstract**

We include in a systematic way the $\eta'$ in baryon chiral perturbation theory. The most general relativistic effective Lagrangian describing the interaction of the lowest lying baryon octet with the Goldstone boson octet and the $\eta'$ is presented up to linear order in the derivative expansion and its heavy baryon limit is obtained. As explicit examples, we calculate the baryon masses and the $\pi N$ $\sigma$-term up to one-loop order in the heavy baryon formulation. A systematic expansion in the meson masses is possible, and appearing divergences are renormalized.

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# 1 Introduction

Chiral perturbation theory is the effective field theory of QCD at very low energies. The QCD Lagrangian with massless quarks exhibits an $SU(3)_R \times SU(3)_L$ chiral symmetry which is broken down spontaneously to $SU(3)_V$, giving rise to a Goldstone boson octet of pseudoscalar mesons which become massless in the chiral limit of zero quark masses. On the other hand, the axial $U(1)$ symmetry of the QCD Lagrangian is broken by the anomaly. The corresponding pseudoscalar singlet would otherwise have a mass comparable to the pion mass [1]. Such a particle is missing in the spectrum and the lightest candidate would be the $\eta'$ with a mass of 958 MeV which is considerably heavier than the octet states. In conventional chiral perturbation theory the $\eta'$ is not included explicitly, although it does show up in the form of a contribution to a coupling coefficient of the Lagrangian, a so-called low-energy constant (LEC).

Experiment suggests that mixing between the Goldstone boson octet and the singlet $\eta_0$ occurs. More precisely, the singlet mixes with the uncharged octet states, $\pi^0$ and $\eta_8$. We will work in the isospin limit of identical light quark masses, $m_u = m_d$, throughout this work. In this case, the $\pi^0$ decouples and only $\eta_0$-$\eta_8$ mixing remains, yielding the physical states $\eta$ and $\eta'$.

In order to include this effect in chiral perturbation theory one should treat the $\eta'$ as a dynamical field variable instead of integrating it out from the effective theory. This approach is also motivated by large $N_c$ considerations. In this limit the axial anomaly is suppressed by powers of $1/N_c$ and gives rise to a ninth Goldstone boson, the $\eta'$.

The purpose of this work is to include the $\eta'$ in baryon chiral perturbation theory in a systematic fashion without invoking large $N_c$ arguments. It will be shown that observables can be expanded in the masses of the meson octet and singlet simultaneously. The relative size of the expansion parameters is given by $m_\eta/m_{\eta'}$. In this introductory presentation we will restrict ourselves to the development of the theory and do not address such issues as determination of the appearing LECs from experiment and convergence of the series.

In the following section, we present the purely mesonic Lagrangian including the $\eta'$. The extension to the baryonic case is discussed in Section 3. As explicit examples we present the calculation of the baryon octet masses and the $\pi N$ $\sigma$-term up to one-loop order. A novel feature of this approach is the appearance of divergences at one-loop level which are renormalized in a chiral invariant way. We conclude with a short summary.

# 2 The mesonic Lagrangian

In this section we will consider the purely mesonic Lagrangian including the $\eta'$. The derivation of this Lagrangian has been given elsewhere, see e.g. [2, 3, 4], so

we will restrict ourselves to the repetition of some of the basic formulae which are needed in the present work. In [3, 4] the topological charge operator coupled to an external field is added to the QCD Lagrangian

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{g^2}{16\pi^2}\theta(x)\mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu}) \tag{1}$$

with $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$ and $\mathrm{tr}_c$ is the trace over the color indices. Under $U(1)_R \times U(1)_L$ the axial $U(1)$ anomaly adds a term $-(g^2/16\pi^2)2N_f\,\alpha\,\mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu})$ to the QCD Lagrangian, with $N_f$ being the number of different quark flavors and $\alpha$ the angle of the global axial $U(1)$ rotation. The vacuum angle $\theta(x)$ is in this context treated as an external field that transforms under an axial $U(1)$ rotation as

$$\theta(x) \rightarrow \theta'(x) = \theta(x) - 2N_f\alpha. \tag{2}$$

Then the term generated by the anomaly in the fermion determinant is compensated by the shift in the $\theta$ source and the Lagrangian from Eq.(1) remains invariant under axial $U(1)$ transformations. The symmetry group $SU(3)_R \times SU(3)_L$ of the Lagrangian $\mathcal{L}_{QCD}$ is extended to $U(3)_R \times U(3)_L$ for $\mathcal{L}$.[3] This property remains at the level of an effective theory and the additional source $\theta$ also shows up in the effective Lagrangian. Let us consider the purely mesonic effective theory first. The lowest lying pseudoscalar meson nonet is summarized in a matrix valued field $U(x)$

$$U(\phi,\eta_0) = u^2(\phi,\eta_0) = \exp\{2i\phi/F_\pi + i\sqrt{\frac{2}{3}}\eta_0/F_0\}, \tag{3}$$

where $F_\pi \simeq 92.4$ MeV is the pion decay constant and the singlet $\eta_0$ couples to the singlet axial current with strength $F_0$. The unimodular part of the field $U(x)$ contains the degrees of freedom of the Goldstone boson octet $\phi$

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K^0} & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \quad , \tag{4}$$

while the phase $\det U(x) = e^{i\sqrt{6}\eta_0/F_0}$ describes the $\eta_0$ . The symmetry $U(3)_R \times U(3)_L$ does not have a dimension-nine irreducible representation and consequently does not exhibit a nonet symmetry. We have therefore used the different notation $F_0$ for the decay constant of the singlet field. The effective Lagrangian is formed with the fields $U(x)$, derivatives thereof and also includes both the quark mass

[3]To be more precise, the Lagrangian changes by a total derivative which gives rise to the Wess-Zumino term. We will neglect this contribution since the corresponding terms involve five or more meson fields which do not play any role for the discussions here.

matrix $\mathcal{M}$ and the vacuum angle $\theta$: $\mathcal{L}_{\text{eff}}(U, \partial U, \ldots, \mathcal{M}, \theta)$. Under $U(3)_R \times U(3)_L$ the fields transform as follows

$$U' = RUL^\dagger \quad , \qquad \mathcal{M}' = R\mathcal{M}L^\dagger \quad , \qquad \theta'(x) = \theta(x) - 2N_f\alpha \tag{5}$$

with $R \in U(3)_R$, $L \in U(3)_L$, but the Lagrangian remains invariant. In order to incorporate the baryons into the effective theory it is convenient to form an object of axial-vector type with one derivative

$$u_\mu = iu^\dagger \nabla_\mu U u^\dagger \tag{6}$$

with $\nabla_\mu$ being the covariant derivative of $U$. The matrix $u_\mu$ transforms under $U(3)_R \times U(3)_L$ as a matter field,

$$u_\mu \to u'_\mu = K u_\mu K^\dagger \tag{7}$$

with $K(U, R, L)$ the compensator field representing an element of the conserved subgroup $U(3)_V$. The phase of the determinant $\det U(x) = e^{i\sqrt{6}\eta_0/F_0}$ transforms under axial $U(1)$ as $\sqrt{6}\eta'_0/F_0 = \sqrt{6}\eta_0/F_0 + 2N_f\alpha$ so that the combination $\sqrt{6}\eta_0/F_0 + \theta$ remains invariant. It is more convenient to replace the variable $\theta$ by this invariant combination, $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \ldots, \mathcal{M}, \sqrt{6}\eta_0/F_0 + \theta)$. One can now construct the effective Lagrangian in these fields that respects the symmetries of the underlying theory. In particular, the Lagrangian is invariant under $U(3)_R \times U(3)_L$ rotations of $U$ and $\mathcal{M}$ at a fixed value of the last argument. The most general Lagrangian up to and including terms with two derivatives and one factor of $\mathcal{M}$ reads

$$\mathcal{L}_\phi = -V_0 + V_1\langle u_\mu u^\mu\rangle + V_2\langle \chi_+\rangle + iV_3\langle \chi_-\rangle + V_4\langle u_\mu\rangle\langle u^\mu\rangle. \tag{8}$$

The expression $\langle \ldots \rangle$ denotes the trace in flavor space and the quark mass matrix $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ enters in the combinations

$$\chi_\pm = 2B_0(u\mathcal{M}u \pm u^\dagger \mathcal{M} u^\dagger) \tag{9}$$

with $B_0 = -\langle 0|\bar{q}q|0\rangle/F_\pi^2$ the order parameter of the spontaneous symmetry violation. Note that a term of the type $\langle u_\mu\rangle \nabla^\mu\theta$ can be transformed away [3] and a term proportional to $\nabla_\mu\theta\nabla^\mu\theta$ does not enter the calculations performed in the present work and will be neglected. The coefficients $V_i$ are functions of the variable $\sqrt{6}\eta_0/F_0 + \theta$, $V_i(\sqrt{6}\eta_0/F_0 + \theta)$, and can be expanded in terms of this variable. At a given order of derivatives of the meson fields $U$ and insertions of the quark mass matrix $\mathcal{M}$ one obtains an infinite string of increasing powers of the singlet field $\eta_0$ with couplings which are not fixed by chiral symmetry. The terms $V_{1,\ldots,4}$ are of second chiral order, whereas $V_0$ is of zeroth chiral order. Parity conservation implies that the $V_i$ are all even functions of $\sqrt{6}\eta_0/F_0 + \theta$ except $V_3$, which is odd, and $V_1(0) = V_2(0) = F_\pi^2/4$ gives the correct normalizaton for the

quadratic terms of the Goldstone boson octet. The kinetic energy of the the $\eta_0$ singlet field obtains contributions from $V_1\langle u_\mu u^\mu\rangle$ and $V_4\langle u_\mu\rangle\langle u^\mu\rangle$ which read

$$\Big(\frac{F_\pi^2}{2F_0^2}+\frac{6}{F_0^2}V_4(0)\Big)\partial_\mu\eta_0\partial^\mu\eta_0. \tag{10}$$

We renormalize the $\eta_0$ field in such a way that the coefficient in brackets is 1/2 in analogy to the kinetic term of the octet. By redefining $F_0$ and keeping for simplicity the same notation both for $\eta_0$ and $F_0$ one arrives at the same Lagrangian as in Eq.(8) but with $V_4(0)=(F_0^2-F_\pi^2)/12$ in order to ensure the usual normalization for the kinetic term of a pseudoscalar particle.

For our considerations here we can safely neglect the source $\theta$. The coefficients $V_i$ are then functions of $\eta_0$ only, $V_i(\eta_0)$, and their Taylor expansions read at lowest orders, e.g.,

$$\begin{aligned} V_0 &= \text{const.} + v\,\eta_0^2+\ldots \\ V_2 &= \frac{1}{4}F_\pi^2 + w\,\eta_0^2+\ldots \\ V_3 &= x\,\eta_0+\ldots \end{aligned} \tag{11}$$

where the ellipses denote terms with higher powers of $\eta_0$. Here, we presented only the terms which enter our calculation. Note that the quadratic term in $V_0$ contributes to the $\eta_0$ mass which does not vanish in the chiral limit, i.e. the $\eta_0$ is not a Goldstone boson.

Expanding the Lagrangian in terms of the meson fields one observes terms quadratic in the meson fields that contain the factor $\eta_0\eta_8$ which leads to $\eta_0$-$\eta_8$ mixing. Such terms arise from the explicitly symmetry breaking terms $V_2\langle\chi_+\rangle+iV_3\langle\chi_-\rangle$ and read

$$-\Big(\frac{2\sqrt{2}}{3}\frac{F_\pi}{F_0}+\frac{8}{\sqrt{3}}\frac{1}{F_\pi}x\Big)B_0(\hat{m}-m_s)\eta_0\eta_8 \tag{12}$$

with $\hat{m}=\frac{1}{2}(m_u+m_d)$. The states $\eta_0$ and $\eta_8$ are therefore not mass eigenstates. The mixing yields the eigenstates $\eta$ and $\eta'$,

$$\begin{aligned} |\eta\rangle &= \cos\theta\,|\eta_8\rangle-\sin\theta\,|\eta_0\rangle \\ |\eta'\rangle &= \sin\theta\,|\eta_8\rangle+\cos\theta\,|\eta_0\rangle, \end{aligned} \tag{13}$$

where we have neglected other pseudoscalar isoscalar states which could mix with both $\eta_0$ and $\eta_8$ and we assume that the mixing parameters do not depend on the energy of the state. The $\eta$-$\eta'$ mixing angle can be determined from the two photon decays of $\pi^0,\eta,\eta'$, which require a mixing angle around -20° [5]. We will make use of this experimental input in order to diagonalize the mass terms of the effective mesonic Lagrangian. Since we work in the isospin limit $m_u=m_d$, this is the only mixing between the meson states.

# 3 Inclusion of baryons

We now proceed by including the lowest lying baryon octet into the effective theory. The baryon octet $B$ is given by the matrix

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} \tag{14}$$

which transforms as a matter field

$$B \rightarrow B' = KBK^\dagger . \tag{15}$$

Up to linear order in the derivative expansion the most general relativistic effective Lagrangian describing the interaction of the baryon octet with the meson nonet reads

$$\begin{aligned} \mathcal{L}_{\phi B} &= iW_1\langle[D^\mu, \bar{B}]\gamma_\mu B\rangle - iW_1^*\langle\bar{B}\gamma_\mu[D^\mu, B]\rangle + W_2\langle\bar{B}B\rangle \\ &\quad + W_3\langle\bar{B}\gamma_\mu\gamma_5\{u^\mu, B\}\rangle + W_4\langle\bar{B}\gamma_\mu\gamma_5[u^\mu, B]\rangle + W_5\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\langle u^\mu\rangle \\ &\quad + W_6\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\nabla^\mu\theta + iW_7\langle\bar{B}\gamma_5 B\rangle \end{aligned} \tag{16}$$

with $D_\mu$ being the covariant derivative of the baryon fields. The $W_i$ are functions of the combination $\sqrt{6}\eta_0/F_0 + \theta$. From parity it follows that they are even in this variable except $W_7$ which is odd. One can further reduce the number of independent terms by making the following transformation. By decomposing the baryon fields into their left- and right handed components

$$B_{R/L} = \frac{1}{2}(1 \pm \gamma_5)B \tag{17}$$

and transforming the left- and right-handed states separately via

$$\begin{aligned} B_{R/L} &\rightarrow \frac{1}{\sqrt{W_2 \pm iW_7}} B_{R/L} \\ \bar{B}_{R/L} &\rightarrow \frac{1}{\sqrt{W_2 \mp iW_7}} \bar{B}_{R/L} \end{aligned} \tag{18}$$

one can eliminate the $\langle\bar{B}\gamma_5 B\rangle$ term and simplify the coefficient of $\langle\bar{B}B\rangle$. The details of this calculation are given in App. A. The Lagrangian in Eq.(16) reduces then to

$$\begin{aligned} \mathcal{L}_{\phi B} &= iU_1\Big(\langle[D^\mu, \bar{B}]\gamma_\mu B\rangle - \langle\bar{B}\gamma_\mu[D^\mu, B]\rangle\Big) - \overset{\circ}{M}\langle\bar{B}B\rangle + U_2\langle\bar{B}\gamma_\mu\gamma_5\{u^\mu, B\}\rangle \\ &\quad + U_3\langle\bar{B}\gamma_\mu\gamma_5[u^\mu, B]\rangle + U_4\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\langle u^\mu\rangle + U_5\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\nabla^\mu\theta \end{aligned} \tag{19}$$

with $\overset{\circ}{M}$ being the baryon octet mass in the chiral limit. The coefficients $U_i$ are real and even functions of $\sqrt{6}\eta_0/F_0 + \theta$. The last term which includes the

derivative of the source $\theta$ can be disregarded for the processes considered here, and we can set $\theta = 0$ for our purposes. The expansion of the coefficients $U_i$ in terms of $\eta_0$ read

$$\begin{aligned}
U_1 &= -\frac{1}{2} + \lambda_1 \eta_0^2 + \ldots \\
U_2 &= -\frac{1}{2} D + \ldots \\
U_3 &= -\frac{1}{2} F + \ldots \\
U_4 &= \lambda_2 + \ldots
\end{aligned} \tag{20}$$

where the ellipses denote higher orders in $\eta_0$ and we have only shown terms that contribute to the baryon masses up to one-loop order. The axial-vector couplings $D$ and $F$ can be determined from semileptonic hyperon decays. A fit to the experimental data delivers $D = 0.80 \pm 0.01$ and $F = 0.46 \pm 0.01$ [6].

The drawback of the relativistic framework including baryons is that due to the existence of a new mass scale, namely the baryon mass in the chiral limit $\overset{\circ}{M}$, there exists no strict chiral counting scheme, i.e. a one-to-one correspondence between the meson loops and the chiral expansion. In order to overcome this problem one integrates out the heavy degrees of freedom of the baryons, similar to a Foldy-Wouthuysen transformation, so that a chiral counting scheme emerges. We will adopt this procedure here in order to write down the heavy baryon formulation of the theory in the presence of the singlet field. Observables can then be expanded simultaneously in the Goldstone boson octet masses and the $\eta'$ mass that does not vanish in the chiral limit. One obtains a one-to-one correspondence between the meson loops and the expansion in their masses and derivatives both for octet and singlet. Thus, even in the presence of the massive $\eta'$ field a strict chiral counting scheme is possible. The purpose of this presentation is to show that such a scheme can be established. Issues as the convergence of the expansion in the $\eta'$ mass which is in size comparable to the baryon octet masses will not be addressed here. Here, we only mention that the large $N_c$ scheme provides some motivation why higher orders in the $\eta'$ mass could be suppressed, since such terms arise from higher orders in $\eta_0$ in the expansions of the coefficients $U_i$ and $V_i$ of the Lagrangian and are suppressed by powers of $1/N_c$. By neglecting the contributions from the singlet and treating $\eta_8$ as $\eta$ one would again arrive at the conventional chiral expansion. The relative size of the expansion parameters is given by $m_\eta/m_{\eta'}$, which is of order $\sqrt{N_c m_s}$ with $N_c$ the number of colors and $m_s \gg \hat{m}$ has been assumed.

The process of integrating out the heavy degrees of freedom of the baryons from the effective theory is rather well known [7] and we only present the result here. A four-velocity $v$ is assigned to the baryons and the heavy baryon

Lagrangian reads to the order we are working

$$\begin{aligned}\mathcal{L}_{\phi B} &= i\langle \bar{B}[v\cdot D, B]\rangle + \lambda_1 \eta_0^2 \Big( -2 \overset{\circ}{M} \langle \bar{B} B\rangle + i\langle [v\cdot D, \bar{B}] B\rangle - i\langle \bar{B}[v\cdot D, B]\rangle \Big) \\ &\quad -D\langle \bar{B} S_\mu \{u^\mu, B\}\rangle - F\langle \bar{B} S_\mu [u^\mu, B]\rangle + 2\lambda_2 \langle \bar{B} S_\mu B\rangle \langle u^\mu\rangle \end{aligned} \tag{21}$$

where we omitted higher powers in the singlet field $\eta_0$ since they do not contribute to the lowest non-analytic contributions for the baryon masses and the $\pi N$ $\sigma$-term. The Dirac algebra simplifies considerably and $2S_\mu = i\gamma_5 \sigma_{\mu\nu} v^\nu$ denotes the Pauli-Lubanski spin vector.

# 4 Baryon masses

In this section we present the calculation of the baryon octet masses within the new framework of heavy baryon chiral perturbation theory including the $\eta'$. In this work our main concern is to include the $\eta'$ in a systematic way and, therefore, we restrict ourselves to the calculation of the one-loop diagrams of the $\eta$ and $\eta'$ with the Lagrangian given in Eq.(21). A complete analysis of the baryon masses would also require the inclusion of explicitly symmetry breaking terms and the $\pi, K$ loops [8, 9]. We will disregard such terms and consider only the lowest non-analytic contributions of the $\eta$ and $\eta'$ to the baryon masses. The contributing one-loop diagrams of the Lagrangian in Eq.(21) are depicted in Figures 1 and 2. In addition to the usual self-energy diagram, Fig. 1, that already enters the calculation in $SU(3)$ chiral perturbation theory, one obtains a tadpole diagram with a singlet loop, Fig. 2. This singlet diagram delivers a constant contribution to all baryon masses and is therefore not observable since it can be absorbed in a redefinition of $\overset{\circ}{M}$. The contributions of the $\eta$ and $\eta'$ fields from Figs. 1 and 2 are given by

$$\begin{aligned}\delta M_B &= -\frac{1}{24\pi F_\pi^2}\alpha_B^2 \left(m_\eta^3 \cos^2\theta + m_{\eta'}^3 \sin^2\theta\right) \\ &\quad -\frac{\sqrt{2}}{24\pi F_0 F_\pi}[D-3\lambda_2]\alpha_B \sin 2\theta \left(m_{\eta'}^3 - m_\eta^3\right) \\ &\quad -\frac{1}{12\pi F_0^2}[D-3\lambda_2]^2 \left(m_\eta^3 \sin^2\theta + m_{\eta'}^3 \cos^2\theta\right) \\ &\quad +\frac{1}{8\pi^2} \overset{\circ}{M} \lambda_1 \Big( \cos^2\theta\, m_{\eta'}^2 \ln\frac{m_{\eta'}^2}{\mu^2} + \sin^2\theta\, m_\eta^2 \ln\frac{m_\eta^2}{\mu^2}\Big) + \Delta \end{aligned} \tag{22}$$

with $\mu$ being the scale introduced in dimesional regularization and a divergent piece from the tadpole

$$\Delta = 4 \overset{\circ}{M} \lambda_1 \Big(m_{\eta'}^2 \cos^2\theta + m_\eta^2 \sin^2\theta\Big) L, \tag{23}$$

with

$$L = \frac{\mu^{d-4}}{16\pi^2}\Big\{\frac{1}{d-4} - \frac{1}{2}[\ln 4\pi + 1 - \gamma_E]\Big\}. \tag{24}$$

Here, $\gamma_E = 0.5772215$ is the Euler-Mascheroni constant. The coefficients $\alpha_B$ are given by

$$\begin{aligned} \alpha_N &= \frac{1}{2}(D-3F), \qquad \alpha_\Lambda = -D, \qquad \alpha_\Sigma = D, \\ \alpha_\Xi &= \frac{1}{2}(D+3F). \end{aligned} \tag{25}$$

The contributions of the other mesons remain unchanged after the inclusion of the singlet field and are given elsewhere, see e.g. [8]. We have expressed our results in terms of the physical states $\eta$ and $\eta'$. The crucial difference with respect to heavy baryon chiral perturbation theory without the singlet is the appearance of the tadpole contribution which delivers terms quadratic in the meson masses $m_\eta$ and $m_{\eta'}$. This seems to violate the observation that the lowest non-analytic pieces start contributing at third chiral order [10], but both the $\eta$ and the $\eta'$ masses contain a piece from the singlet field which does not vanish in the chiral limit. The $\eta'$ mass and the logarithm thereof can be expanded around this value which leads to a series analytic in the quark masses, whereas for the $\eta$ we note that the mixing angle is of chiral order $\mathcal{O}(p^2)$ for small quark masses. Therefore, Eq.(22) does not contradict the results from [10].

The tadpole is divergent and has to be renormalized. This is a new feature at leading order which did not occur in conventional heavy baryon chiral perturbation theory. Its divergent piece can be renormalized by redefining both the baryon mass in the chiral limit $\overset{\circ}{M}$ and the coefficient $b_0$ of the $SU(3)$ invariant piece of the explicitly symmetry breaking terms, $\langle \bar{B}B\rangle\langle\chi_+\rangle$. To this end, one uses the relation between the masses of the physical states $\eta$ and $\eta'$ and the coefficient of the mass term for $\eta_0$ in $\mathcal{L}_\phi$:

$$m^2_{\eta'}\cos^2\theta + m^2_\eta \sin^2\theta = \Big(\frac{2F_\pi^2}{3F_0^2} - 8w + 8\sqrt{\frac{2}{3}}\frac{x}{F_0}\Big)\, B_0\,(2\hat{m}+m_s) + 2v \tag{26}$$

with $w, v$ and $x$ being parameters of the mesonic Lagrangian as given in Eq.(11). The last term without the quark masses is absorbed by $\overset{\circ}{M}$, whereas a redefinition of $b_0$ renormalizes the quark mass dependent divergences.

$$\overset{\circ}{M} \;\rightarrow\; \overset{\circ}{M}{}^r - 8\,\overset{\circ}{M}\,\lambda_1 v L \tag{27}$$

$$b_0 \;\rightarrow\; b_0^r + \overset{\circ}{M}\,\lambda_1\Big(\frac{2F_\pi^2}{3F_0^2} - 8w + 8\sqrt{\frac{2}{3}}\frac{x}{F_0}\Big)L. \tag{28}$$

But before proceeding, it is worthwhile taking a closer look at the contributions of the $\eta'$ to $M_\Lambda$ and $M_\Sigma$. The $SU(3)$-breaking contributions of the $\eta'$ read

for these two cases

$$-\frac{1}{24\pi F_\pi^2} D^2\, m_{\eta'}^3 \sin^2\theta \pm \frac{\sqrt{2}}{24\pi F_0 F_\pi}[D-3\lambda_2]D\,\sin 2\theta\, m_{\eta'}^3. \tag{29}$$

Inserting the values $m_{\eta'} = 958$ MeV and $\theta = -20°$ and using the central values for $D$ and $F$ from [6][4], both contributions are relatively small only if $D \simeq 3\lambda_2$ yielding a mass shift of $-103$ MeV in both cases. For all other values of $\lambda_2$ one obtains substantial $SU(3)$ breaking contributions of the $\eta'$ to the baryon masses. In order to get a rough estimate we set $\lambda_2$ equal to zero and use $F_0 = F_\pi = 92.4$ MeV. From Eq.(29) one obtains the numerical values $-907$ MeV and 701 MeV for the $\Lambda$ and $\Sigma$ mass shifts, respectively. A more quantitative statement about these contributions can only be made if one has a reliable estimate of the parameter $\lambda_2$, but this is beyond the scope of this work.

# 5 The $\pi N$ $\sigma$-term

Closely related to the nucleon mass is the $\pi N$ $\sigma$-term

$$\sigma_{\pi N}(t) = \hat{m}\langle p'|\bar{u}u + \bar{d}d|p\rangle \tag{30}$$

with $|p\rangle$ a proton state with momentum $p$ and $t = (p'-p)^2$ the momentum transfer squared. The $\sigma$-term vanishes in the chiral limit of zero quark masses and measures the scalar quark density inside the proton. Thus it is particularly suited to test our understanding of spontaneous and explicit chiral symmetry breaking. At zero momentum transfer squared, $t = 0$, the $\pi N$ $\sigma$-term is related to the nucleon mass $M_N$ via the Feynman-Hellmann theorem

$$\sigma_{\pi N}(0) = \hat{m}\frac{\partial M_N}{\partial \hat{m}}. \tag{31}$$

Again we will restrict ourselves to the presentation of $\eta$ and $\eta'$ loops. Contributions from the contact terms of second chiral order and $\pi, K$ loops have already been calculated in [8]. The result is

$$\begin{aligned}
\sigma_{\pi N}(0) \;=\; & -\frac{1}{64\pi F_\pi^2}[D-3F]^2\, m_\pi^2 \left(\mathcal{A}\sin^2\theta\, m_{\eta'} + \mathcal{B}\cos^2\theta\, m_\eta\right) \\
& -\frac{1}{8\pi F_0^2}[D-3\lambda_2]^2\, m_\pi^2 \left(\mathcal{A}\cos^2\theta\, m_{\eta'} + \mathcal{B}\sin^2\theta\, m_\eta\right) \\
& +\frac{\sqrt{2}}{32\pi F_0 F_\pi}[D-3\lambda_2][D-3F]\, m_\pi^2\, \sin 2\theta \left(\mathcal{A}m_{\eta'} - \mathcal{B}m_\eta\right) \\
& +\frac{1}{8\pi^2}\, \overset{\circ}{M}\, \lambda_1 m_\pi^2 \left(\mathcal{A}\cos^2\theta\, [1+\ln\frac{m_{\eta'}^2}{\mu^2}] + \mathcal{B}\sin^2\theta\, [1+\ln\frac{m_\eta^2}{\mu^2}]\right) + \Delta'
\end{aligned} \tag{32}$$

[4]We assume here that the inclusion of the $\eta'$ does not alter the values for $D$ and $F$ significantly.

with the coefficients

$$\mathcal{A} = \frac{1}{3}\sin^2\theta + \Big(\frac{2F_\pi^2}{3F_0^2} - 8w + 8\sqrt{\frac{2}{3}}\frac{x}{F_0}\Big)\cos^2\theta + \Big(\frac{\sqrt{2}F_\pi}{3F_0} + 4\frac{x}{\sqrt{3}F_\pi}\Big)\sin 2\theta \tag{33}$$

$$\mathcal{B} = \frac{1}{3}\cos^2\theta + \Big(\frac{2F_\pi^2}{3F_0^2} - 8w + 8\sqrt{\frac{2}{3}}\frac{x}{F_0}\Big)\sin^2\theta - \Big(\frac{\sqrt{2}F_\pi}{3F_0} + 4\frac{x}{\sqrt{3}F_\pi}\Big)\sin 2\theta \tag{34}$$

and a divergent piece

$$\Delta' = 4 \overset{\circ}{M} \lambda_1 m_\pi^2 \Big(\mathcal{A}\cos^2\theta + \mathcal{B}\sin^2\theta\Big) L. \tag{35}$$

This time the divergent part is renormalized by the term $\langle \bar{B}B\rangle\langle\chi_+\rangle$ only, since $\overset{\circ}{M}$ does not contribute to the $\sigma$-term. Using the same renormalization prescription for $b_0$ as in Eq.(28) one achieves cancellation of the divergences.

Of particular interest is the shift of the $\sigma$-term to the Cheng-Dashen point $t = 2m_\pi^2$. It is convenient to work in the Breit frame, $v\cdot p = v\cdot p'$. The $\eta$ and $\eta'$ contributions to the $\sigma$-term shift read

$$\begin{aligned}
&\sigma_{\pi N}(2m_\pi^2) - \sigma_{\pi N}(0) = \\
&-\frac{1}{8\pi} m_\pi^2 \Big(\frac{1}{24F_\pi^2}[D-3F]^2\sin^2\theta + \frac{1}{3F_0^2}[D-3\lambda_2]^2\cos^2\theta \\
&-\frac{\sqrt{2}}{12F_0F_\pi}[D-3\lambda_2][D-3F]\sin 2\theta\Big)\mathcal{A}\Big(-m_{\eta'} + \frac{m_{\eta'}^2 - m_\pi^2}{\sqrt{2}m_\pi}\ln\frac{\sqrt{2}m_{\eta'}+m_\pi}{\sqrt{2}m_{\eta'}-m_\pi}\Big) \\
&-\frac{1}{8\pi} m_\pi^2 \Big(\frac{1}{24F_\pi^2}[D-3F]^2\cos^2\theta + \frac{1}{3F_0^2}[D-3\lambda_2]^2\sin^2\theta \\
&+\frac{\sqrt{2}}{12F_0F_\pi}[D-3\lambda_2][D-3F]\sin 2\theta\Big)\mathcal{B}\Big(-m_\eta + \frac{m_\eta^2 - m_\pi^2}{\sqrt{2}m_\pi}\ln\frac{\sqrt{2}m_\eta+m_\pi}{\sqrt{2}m_\eta-m_\pi}\Big) \\
&+\frac{1}{4\pi^2}\overset{\circ}{M}\lambda_1 m_\pi^2\Big(\mathcal{A}\cos^2\theta\Big[-1+\sqrt{\frac{2m_{\eta'}^2}{m_\pi^2}-1}\ \arcsin\frac{m_\pi}{\sqrt{2}m_{\eta'}}\Big] \\
&\qquad +\mathcal{B}\sin^2\theta\Big[-1+\sqrt{\frac{2m_\eta^2}{m_\pi^2}-1}\ \arcsin\frac{m_\pi}{\sqrt{2}m_\eta}\Big]\Big).
\end{aligned} \tag{36}$$

In order to give some estimate on the numerical size of the new terms involving the $\eta'$ as compared to conventional $SU(3)\times SU(3)$ chiral perturbation theory, we consider the two cases $\lambda_2 = 0$ and $\lambda_2 = D/3$ with vanishing $w, x, \lambda_1$. One obtains 0.21 MeV and 0.02 MeV for the first and second case, respectively. The

general expression of the $\eta$ and $\eta'$ contributions to the $\sigma$-term shift reads, in units of MeV,

$$\begin{aligned}\sigma_{\pi N}(2m_\pi^2) - \sigma_{\pi N}(0) &= 0.2 - 1.8w + 15.3x + \lambda_2(-1.2 + 14.1w - 106.9x) \\ &\quad +\lambda_2^2(1.9 - 28.1w + 199.3x). \end{aligned} \tag{37}$$

The tadpole contribution proportional to $\lambda_1$ is for realistic values of the parameters about $10^{-3}$ MeV and has been neglected here. On the other hand, the $\sigma$-term shift depends significantly on the other parameters, particularly $x$. It would be desirable to have a reliable estimate on these parameters in order to make a more quantitative statement about the $\eta'$ contributions to the $\sigma$-term shift. These results can be compared with the $\eta$ contribution of about 0.01 MeV from a calculation without the $\eta'$ in conventional chiral perturbation theory. Note, however, that this number is only a small fraction of a calculation including both pion and kaon loops. The $\sigma$-term shift to the Cheng-Dashen point is dominated by the pion loop contribution and is at one-loop order about 7.5 MeV [8]. The $\sigma$-term for general $t$ is given in App. B.

# 6 Summary

In this work we included in a systematic way the $\eta'$ in baryon chiral perturbation theory. After setting up the most general relativistic Lagrangian to first order in the derivative expansion we derived its heavy baryon limit. In the heavy baryon formulation a one-to-one correspondence between the number of octet and singlet meson loops and the expansion in the pertinent masses emerges. The relative size of the expansion parameters is given by $m_\eta/m_{\eta'}$.

As explicit examples we presented the calculation of the baryon masses and the $\pi N$ $\sigma$-term up to one loop order in this new framework. In the case of the baryon masses it turns out that there are sizeable contributions of the $\eta'$, unless a certain combination of LECs not fixed by chiral symmetry happens to be small.

A novel feature of this approach is the appearance of a tadpole diagram at leading order in the expansion which delivers a divergence. The divergent piece can be compensated by redefining some of the low-energy constants. This work introduces the $\eta'$ as a dynamical field variable without invoking large $N_c$ arguments. Other issues such as the convergence of the expansion in the meson masses or the connection to large $N_c$ baryon chiral perturbation theory will be addressed in future work.

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# A

In this appendix, we present the calculation which reduces the Lagrangian of Eq.(16) to the one given in Eq.(19). The starting point is the relativistic Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\phi B} \;=\;& iW_1\langle[D^\mu,\bar{B}]\gamma_\mu B\rangle - iW_1^*\langle\bar{B}\gamma_\mu[D^\mu,B]\rangle + W_2\langle\bar{B}B\rangle \\
&+W_3\langle\bar{B}\gamma_\mu\gamma_5\{u^\mu,B\}\rangle + W_4\langle\bar{B}\gamma_\mu\gamma_5[u^\mu,B]\rangle + W_5\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\langle u^\mu\rangle \\
&+W_6\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\nabla^\mu\theta + iW_7\langle\bar{B}\gamma_5 B\rangle. \qquad\text{(A.1)}
\end{aligned}
$$

By decomposing the baryon fields into their left- and right-handed components

$$
B_{R/L} = \frac{1}{2}(1\pm\gamma_5)B \qquad\text{(A.2)}
$$

and transforming the left- and right-handed states separately via

$$
\begin{aligned}
B_{R/L} &\to \frac{1}{\sqrt{W_2\pm iW_7}}B_{R/L} \\
\bar{B}_{R/L} &\to \frac{1}{\sqrt{W_2\mp iW_7}}\bar{B}_{R/L} \qquad\text{(A.3)}
\end{aligned}
$$

the single terms of the Lagrangian transform as follows:

$$
W_2\langle\bar{B}B\rangle + iW_7\langle\bar{B}\gamma_5 B\rangle \quad\to\quad \langle\bar{B}B\rangle; \qquad\text{(A.4)}
$$

$$
\begin{aligned}
& iW_1\langle[D^\mu,\bar{B}]\gamma_\mu B\rangle - iW_1^*\langle\bar{B}\gamma_\mu[D^\mu,B]\rangle \quad\to \\
&\qquad \frac{i}{2}\frac{W_1+W_1^*}{\sqrt{W_2^2+W_7^2}}\Big(\langle[D^\mu,\bar{B}]\gamma_\mu B\rangle - \langle\bar{B}\gamma_\mu[D^\mu,B]\rangle\Big) \\
&\qquad +\frac{W_1+W_1^*}{2\,[W_2^2+W_7^2]^{3/2}}(W_7\partial_\mu W_2 - W_2\partial_\mu W_7)\langle\bar{B}\gamma_\mu\gamma_5 B\rangle, \qquad\text{(A.5)}
\end{aligned}
$$

whereas the terms $W_3$ to $W_6$ remain invariant under this transformation. The last term in Eq.(A.5) can be absorbed into $W_5$ and $W_6$. We rescale the baryon fields in such a way that at lowest order the coefficient of the kinetic term is $-1/2$. From matching to conventional $SU(3)\times SU(3)$ baryon chiral perturbation theory it follows that the coefficient of the mass term $\langle\bar{B}B\rangle$ is $-\overset{\circ}{M}$ and we arrive at the Lagrangian given in Eq.(19).

# B

For general $t$ the contributions of the $\eta$ and $\eta'$ to the $\pi N$ $\sigma$-term are given by

$$
\sigma_{\pi N}(t) =
$$

$$
\begin{aligned}
&-\frac{1}{8\pi}\, m_\pi^2 \Big(\frac{1}{24F_\pi^2}[D-3F]^2 \sin^2\theta + \frac{1}{3F_0^2}[D-3\lambda_2]^2\cos^2\theta \\
&-\frac{\sqrt{2}}{12F_0F_\pi}[D-3\lambda_2][D-3F]\sin 2\theta\Big)\,\mathcal{A}\Big(2m_{\eta'} + \frac{m_{\eta'}^2-\frac{1}{2}t}{\sqrt{t}}\ln\frac{2m_{\eta'}+\sqrt{t}}{2m_{\eta'}-\sqrt{t}}\Big) \\
&-\frac{1}{8\pi}\, m_\pi^2 \Big(\frac{1}{24F_\pi^2}[D-3F]^2 \cos^2\theta + \frac{1}{3F_0^2}[D-3\lambda_2]^2\sin^2\theta \\
&+\frac{\sqrt{2}}{12F_0F_\pi}[D-3\lambda_2][D-3F]\sin 2\theta\Big)\,\mathcal{B}\Big(2m_{\eta} + \frac{m_{\eta}^2-\frac{1}{2}t}{\sqrt{t}}\ln\frac{2m_{\eta}+\sqrt{t}}{2m_{\eta}-\sqrt{t}}\Big) \\
&+\frac{1}{8\pi^2}\,\overset{\circ}{M}\,\lambda_1 m_\pi^2\Big(\mathcal{A}\cos^2\theta\,\Big[-1+\ln\frac{m_{\eta'}^2}{\mu^2}+2\sqrt{\frac{4m_{\eta'}^2}{t}-1}\ \arcsin\frac{\sqrt{t}}{2m_{\eta'}}\Big] \\
&\qquad\qquad +\mathcal{B}\sin^2\theta\,\Big[-1+\ln\frac{m_{\eta}^2}{\mu^2}+2\sqrt{\frac{4m_{\eta}^2}{t}-1}\ \arcsin\frac{\sqrt{t}}{2m_{\eta}}\Big]\Big)+\Delta'
\end{aligned}
\tag{B.1}
$$

with the divergence

$$
\Delta' = 4\,\overset{\circ}{M}\,\lambda_1 m_\pi^2\Big(\mathcal{A}\cos^2\theta+\mathcal{B}\sin^2\theta\Big)\,L. \tag{B.2}
$$

# References


[1] S. Weinberg, Phys. Rev. **D11** (1975) 3583

[2] J. Gasser, H. Leutwyler, Nucl. Phys. **B250** (1985) 465

[3] H. Leutwyler, Phys. Lett. **B374** (1996) 163

[4] P. Herrera-Siklódy et al., Nucl. Phys. **B497** (1997) 345

[5] Particle Data Group, C. Caso et al., Eur. Phys. J **C3** (1998) 1

[6] F. E. Close, R. G. Roberts, Phys. Lett. **B316** (1993) 165;
B. Borasoy, Phys. Rev. **D59** (1999) 054021

[7] T. Mannel, W. Roberts, G. Thompson, Nucl. Phys. **B368** (1992) 204

[8] V. Bernard, N. Kaiser, U.-G. Meißner, Z. Phys. **C60** (1993) 111

[9] B. Borasoy, U.-G. Meißner, Ann. Phys. **254** (1997) 192

[10] J. Gasser, A. Zepeda, Nucl. Phys. **B174** (1980) 445

## Figure captions

Fig.1 Shown is the self-energy diagram for the baryon octet. Solid and dashed lines denote baryons and pseudoscalar mesons, respectively.

Fig.2 Tadpole diagram contributing at leading order to the baryon masses. Solid and dashed lines denote the baryons and the meson singlet, respectively.

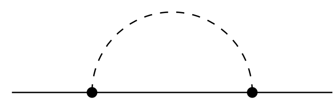

Figure 1

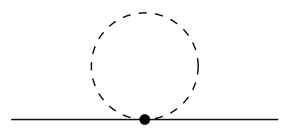

Figure 2